Appendix

In order to establish a correct formulae for the critical values of the Mann-Whitney test the following number theoretical question is important: How many partitions of a natural number a exist such that

$$a = a_1 + a_2 + \dots a_{n_1}, \quad 0 \le a_1 \le \dots \le a_{n_1} \le n_2$$

where n_1 and n_2 are given?

We will denote the number of those partitions as $p_{n_1,n_2}(a)$ with $p_{n_1,n_2}(a) = 0$ if $a < 0$ or $a > n_1 n_2$. Because

$$p_{n_1,n_2}(a) = p_{n_2,n_1}(a) \tag{1}$$

(see [*Ostmann* (1956), p. 32ff.]) we can restrict ourselves to the case $n_1 \le n_2$. A simple modification of this proof shows that also $p_{n_1,n_2}(a) = p_{n_1,n_2}(n_1 \cdot n_2 - a)$ is true.

One of the first recursion formulaes was already known to Mann and Whitney:

$$p_{n_1,n_2}(a) = p_{n_1-1,n_1}(a - n_2) + p_{n_1,n_2-1}(a), \tag{2}$$

but this recursion involves partitions with different numbers n_1 and n_2. Our aim is to establish a formulae using only $p_{n_1,n_2}(a)$ with different a.

To this end we consider the generating function on $\mathbb{R}$

$$F_{n_1,n_2}(z) := \sum_{a=0}^{n_1 n_2} p_{n_1,n_2}(a) z^a, \quad |z| < 1. \tag{3}$$

Since $p_{n_1,n_2}(a)$ is equal to the number of partitions we immediately have another equation for the generating function

$$F_{n_1,n_2}(z) := \sum_{a_1=0}^{n_2} \sum_{a_2=a_1}^{n_2} \dots \sum_{a_{n_1}=a_{n_1-1}}^{n_2} z^{a_1+a_2+\dots a_{n_1}}.$$

Our first observation is the fact that the generating function can be written as a product.

Lemma 1 *We have*

$$F_{n_1,n_2}(z) = \prod_{v=1}^{n_1} \left(\frac{1 - z^{n_2+v}}{1 - z^v} \right) \tag{4}$$

PROOF: We prove the claim by induction over n_1. Because of (1) we start with $n_1 = 1$:

$$\sum_{a=0}^{n_2} p_{1,n_2}(a) z^a = \sum_{a=0}^{n_2} z^a = \frac{1 - z^{n_2+1}}{1 - z}.$$

The induction step uses the well known recursion (2):

$$\begin{aligned}
\sum_{a=0}^{n_1 n_2} p_{n_1,n_2}(a) z^a &= z^{n_2} \sum_{a=0}^{n_1 n_2} p_{n_1-1,n_2}(a-n_2) z^{a-n_2} + \sum_{a=0}^{n_1 n_2} p_{n_1,n_2-1}(a) z^a \\
&= z^{n_2} \prod_{v=1}^{n_1-1} \left(\frac{1-z^{n_2+v}}{1-z^v} \right) + \prod_{v=1}^{n_1} \left(\frac{1-z^{n_2-1+v}}{1-z^v} \right) \\
&= \prod_{v=1}^{n_1} \left(\frac{1-z^{n_2+v}}{1-z^v} \right)
\end{aligned}$$

which was to be shown. ■

In order to develop another recursion formulae we define the following function

$$\sigma(n; n_1, n_2) := \sum_{n \bmod d = 0} \varepsilon_d d \quad \text{where } \varepsilon_d = \begin{cases} 1, & \text{where } 1 \le d \le n_1, \\ 0, & \text{else}, \\ -1, & \text{where } n_2 + 1 \le d \le n_2 + n_1. \end{cases}$$

Then, the following holds.

Lemma 2 *We have*

$$a p_{n_1,n_2}(a) = \sum_{i=0}^{a-1} p_{n_1,n_2}(i) \sigma(a-i; n_1, n_2).$$

PROOF: In order to verify the proposition we start with

$$\begin{aligned}
\frac{F'_{n_1,n_2}(z)}{F_{n_1,n_2}(z)} &= \frac{d}{dz} \left(\ln F_{n_1,n_2}(z) \right) \\
&= \sum_{v=1}^{n_1} \sum_{m=1}^{\infty} \left(v z^{mv-1} - (n_2+v) z^{(n_2+v)m-1} \right) \\
&= \sum_{n=1}^{\infty} \left(\left(\sum_{vm=n,\ 1 \le v \le n_1} z^{n-1} \right) - \left(\sum_{vm=n,\ n_2+1 \le v \le n_2+n_1} z^{n-1} \right) \right) \\
&= \sum_{n=1}^{\infty} \sigma(n; n_1, n_2) z^{n-1}.
\end{aligned}$$

But this implies

$$F'_{n_1,n_2}(z) = F_{n_1,n_2}(z) \sum_{n=1}^{\infty} \sigma(n; n_1, n_2) z^{n-1}$$

or

$$\sum_{a=1}^{n_1 n_2} a p_{n1,n_2}(a) z^{a-1} = \sum_{i=0}^{n_1 n_2} p_{n1,n_2}(i) \sigma(i; n_1, n_2) z^{a+i-1}$$

which in turn gives the required equation by comparison of coefficients. ■

References

[*Ostmann* (1956)] Ostmann, H.-H. (1956) *Additive Zahlentheorie* [in German], Berlin Heidelberg Göttingen.